

ON SOLVABILITY OF THE AUTOMORPHISM GROUP OF A FINITE-DIMENSIONAL ALGEBRA

ALEXANDER PEREPECHKO

ABSTRACT. Consider an automorphism group of a finite-dimensional algebra. S. Halperin conjectured that the unity component of this group is solvable if the algebra is a complete intersection. The solvability criterion recently obtained by M. Schulze [12] provides a proof to a local case of this conjecture as well as gives an alternative proof of S.S.–T. Yau’s theorem [16] based on a powerful result due to G. Kempf. In this note we finish the proof of Halperin’s conjecture and study the extremal cases in Schulze’s criterion, where the algebra of derivations is non-solvable. This allows us to reduce a direct, self-contained proof of Yau’s theorem.

1. INTRODUCTION

Let \mathbb{K} be an algebraically closed field of characteristic zero. We denote by R the algebra of formal power series $\mathbb{K}[[x_1, \dots, x_n]]$ and by \mathfrak{m} the maximal ideal $(x_1, \dots, x_n) \triangleleft R$. Let $I \subset \mathfrak{m}$ be such that $S = R/I$ is a finite-dimensional (or Artin) local algebra with the maximal ideal $\bar{\mathfrak{m}} = \mathfrak{m}/I$.

Consider the automorphism group $\text{Aut } S$. This is an affine algebraic group with the tangent algebra being the Lie algebra of derivations $\text{Der } S$; see [11, Ch.1, §2.3, ex. 2]. So the solvability of the connected component of unity $(\text{Aut } S)^\circ$ (or *unity component* for short) is equivalent to the solvability of the Lie algebra $\text{Der } S$.

In 2009 M. Schulze obtained the following criterion, which has several applications discussed below.

Theorem 1.1 (Schulze, [12]). *Let $S = R/I$ be a finite-dimensional local algebra, where $I \subset \mathfrak{m}^l$. If the inequality*

$$(1.1) \quad \dim(I/\mathfrak{m}I) < n + l - 1$$

holds, then the algebra of derivations $\text{Der } S$ is solvable.

This article provides a generalization of that criterion for a non-local case, see Corollary 4.3, as well as presents a new criterion based on similar techniques, see Theorem 1.12. These two criteria work for different types of algebras.

To mention some applications of Schulze’s criterion let us consider a *regular sequence* $f_1, \dots, f_n \in \mathbb{K}[x_1, \dots, x_n]$ ¹. Equivalently, the quotient $S = \mathbb{K}[x_1, \dots, x_n]/(f_1, \dots, f_n)$ is non-trivial and finite-dimensional and is called a global *complete intersection*.

¹i.e. the image of f_i in the quotient $\mathbb{K}[x_1, \dots, x_n]/(f_1, \dots, f_{i-1})$ is not a zero divisor for all i .

Conjecture 1.2 (Halperin, 1987²). *Suppose that a finite-dimensional algebra S is a global complete intersection. Then the unity component $(\operatorname{Aut} S)^\circ$ of the automorphism group of S is solvable.*

Just a few months later H. Kraft and C. Procesi proved the conjecture in the case of homogeneous polynomials.

Theorem 1.3 (Kraft–Procesi, [9]). *Assuming $\mathbb{K} = \mathbb{C}$, let $f_1, \dots, f_n \in \mathbb{K}[x_1, \dots, x_n]$ be homogeneous polynomials, and the algebra*

$$(1.2) \quad S = \mathbb{K}[x_1, \dots, x_n]/(f_1, \dots, f_n)$$

be finite-dimensional. Then the unity component $(\operatorname{Aut} S)^\circ$ is solvable.

In this case the algebra S is local. Thereby, the generalization of Theorem 1.3 turns out to be a direct consequence of Schulze’s Theorem 1.1 as follows.

Corollary 1.4 (Schulze, [12, Corollary 2]). *Given a local complete intersection $S = R/(f_1, \dots, f_n)$, the group $(\operatorname{Aut} S)^\circ$ is solvable.*

Proof. We may suppose that $f_i \in \mathfrak{m}^2$. Then $\dim(I/\mathfrak{m}I) = n$ and so (1.1) is fulfilled. \square

In Section 4 we introduce a strict criterion for the solvability of the algebra of derivations $\operatorname{Der} S$ for a non-local finite-dimensional algebra S , see Theorem 4.2. This allows us to deduce the global case of Conjecture 1.2 from the local one, thus to finish its proof.

Now let us consider isolated hypersurface singularities (or IHS, for short). Let $p \in \mathbb{K}[x_1, \dots, x_n]$ be such that the hypersurface $\{p = 0\} \subset \mathbb{K}^n$ has an isolated singularity $H = (\{p = 0\}, 0)$ at the origin. Let $J(p) = \left(\frac{\partial p}{\partial x_1}, \dots, \frac{\partial p}{\partial x_n}\right)$ be a *Jacobian ideal* of p . The quotient $A(H) = \mathbb{K}[[x_1, \dots, x_n]]/(p, J(p))$ is called a *local algebra* or a *moduli algebra* of the IHS H .

Since the formal power series ring is local, the algebra $A(H)$ is local as well. There is an analogue of the Hilbert Nullstellensatz for the germs of analytic functions (called *Rückert Nullstellensatz*, see [7, Theorem 3.4.4], [1, 30.12], [13]), which holds for formal power series as well, since there is a purely algebraic proof. So, the radical $\sqrt{(p, J(p))}$ coincides with the maximal ideal \mathfrak{m} . Thus, the ideal $(p, J(p))$ contains some degree of the maximal ideal or, equivalently, the algebra $A(H)$ is finite-dimensional. Conversely, if the algebra $A(H)$ is finite-dimensional then the singularity H is isolated. Indeed, the finite dimensionality of A is equivalent to inclusion $\mathfrak{m}^r \subset I$ for some r , or $\sqrt{(p, J(p))} = \mathfrak{m}$. It implies $\mathbb{V}(p, J(p)) = 0$, and H is the only singularity in some neighbourhood of zero.

It has been proven by J. Mather and S. S.–T. Yau in [10] that two IHS are biholomorphically equivalent if and only if their moduli algebras are isomorphic. Thus, the finite-dimensional local algebra $A(H)$ defines the IHS H up to an analytic isomorphism.

In order to determine when a finite-dimensional local algebra is a moduli algebra of some IHS, S.S.–T. Yau [15] introduced a Lie algebra of derivations $L(H) = \operatorname{Der} A(H)$ called sometimes a *Yau algebra* and obtained the following result.

²This conjecture was proposed by S. Halperin at the conference in honor of J.–L. Koszul.

Theorem 1.5 (Yau,[16]). *The algebra $L(H)$ is solvable.*

Note that generally the Yau algebra does not uniquely determine its moduli algebra. But for *simple* singularities this property holds with only one exception. Their classification is well known and consists of two infinite series A_k, D_k and three exceptional singularities E_6, E_7, E_8 ; e.g. see [2, Chapter 2]. A. Elashvili and G. Khimshiashvili proved the following fact.

Theorem 1.6 (Elashvili–Khimshiashvili, [6, Theorem 3.1]). *Let H_1 and H_2 be two simple IHS, except the pair A_6 and D_5 . Then $L(H_1) \cong L(H_2)$ if and only if H_1 and H_2 are analytically isomorphic.*

Remark 1.7. Assume the polynomial p is quasi-homogeneous, i.e.

$$(1.3) \quad p(\lambda^{k_1}x_1, \dots, \lambda^{k_n}x_n) = \lambda^k p(x_1, \dots, x_n) \text{ for some fixed } k, k_1, \dots, k_n \in \mathbb{N}.$$

Then $p \in J(p)$ and the moduli algebra $\mathbb{K}[[x_1, \dots, x_n]]/(p, J(p))$ is a complete intersection. Under this assumption Theorem 1.5 is a particular case of Corollary 1.4.

In [12] M. Schulze deduces Theorem 1.5 from his criterion. In order to prove it he uses the following deep result of G. Kempf.

Theorem 1.8 (Kempf, [8, Theorem 13]). *Let p be a homogeneous polynomial of degree $d \geq 3$ defined as a regular function on the space \mathbb{C}^n endowed with a linear action of a semisimple group G . If the Jacobian $J(p)$ is a G -invariant subspace then there exists such a G -invariant polynomial q that $J(p) = J(q)$.*

Definition 1.9. Let us call the finite-dimensional local algebra S an *extremal algebra* if the equality $\dim I/\mathfrak{m}I = l + n - 1$ holds.

The description of the extremal algebras with a non-solvable algebra of derivations allows to deduce Theorem 1.5 from Schulze’s criterion without using the Kempf result, as explained in Section 3.

Definition 1.10. Let us say that a graded local finite-dimensional algebra $S = R/I$ is *narrow* if there holds an inequality

$$(1.4) \quad \dim I_k - \dim(\bar{\mathfrak{m}}I)_k \leq k \text{ for all } k = 1, 2, \dots,$$

where J_k is the k th graded component of a graded ideal J . In other words, there exists such a set of homogeneous generators of I that the number of generators of degree k is not greater than k for each k .

Remark 1.11. If $I \subset \mathfrak{m}^r$ then for an algebra $S = R/I$ to be narrow it is sufficient to check the inequality for $k \leq r$, since $I_k = (\mathfrak{m}I)_k$ for $k > r$.

Recall that an *associated graded algebra* of the local algebra S is the algebra $\text{gr } S = \mathbb{K} \oplus (\bar{\mathfrak{m}}/\bar{\mathfrak{m}}^2) \oplus (\bar{\mathfrak{m}}^2/\bar{\mathfrak{m}}^3) \oplus \dots$, i.e. $(\text{gr } S)_i = \bar{\mathfrak{m}}^i/\bar{\mathfrak{m}}^{i+1}$. Now introduce a solvability criterion as follows.

Theorem 1.12. *Suppose that the associated graded algebra $\text{gr } S$ of a local finite-dimensional algebra S is narrow. Then the algebra of derivations $\text{Der } S$ is solvable.*

The proof is given below. Finally, in the last section we give a lower bound for the dimension of the automorphism group and obtain an algebra with the unipotent automorphism group.

Let us mention a related result on solvability of the group of equivariant automorphisms. Consider a connected affine algebraic group G and an irreducible affine G -variety X . Assume that the number of G -orbits on X is finite and X contains a G -fixed point. Then the unity component $(\text{Aut}_G X)^\circ$ of the group of G -equivariant automorphisms of the variety X is solvable; see [3, Theorem 1].

2. SOLVABILITY CRITERIA

In this section we provide a simplified proof of Theorem 1.1 and introduce a new solvability criterion.

If $I \supset \mathfrak{m}^r \triangleleft \mathbb{K}[x_1, \dots, x_n]$ for some r then $R/\tilde{I} \cong \mathbb{K}[x_1, \dots, x_n]/I$, where \tilde{I} is an ideal in the algebra of formal power series generated by I . Therefore it makes no difference whether the local algebra is obtained by factorization of the algebra of polynomials or the algebra of formal power series.

Proposition 2.1. *Suppose that the ideal $I \triangleleft R$ is represented in the form $I = W \oplus \mathfrak{m}I$. Then the ideal (W) generated by subspace W coincides with I .*

Proof. Consider the factorization mapping $\varphi: R \rightarrow R/(W)$. The quotient algebra is a local algebra with the maximal ideal $\varphi(\mathfrak{m})$. The decomposition $I = W \oplus \mathfrak{m}I$ implies $\varphi(I) = \varphi(\mathfrak{m}I)$. Since the ring R is Noetherian, the ideals I and $\varphi(I)$ are finitely generated. Then by Nakayama's Lemma (see [4, Proposition 2.6]) there holds $\varphi(I) = 0$, i.e. $(W) = I$. \square

Corollary 2.2. *The minimal number of generators of the ideal I is equal to $\dim W = \dim(I/\mathfrak{m}I)$.*

Note that Proposition 2.1 does not hold for the algebra of polynomials. For example take an ideal $I = \mathfrak{m}^2 \triangleleft \mathbb{K}[x]$ and decompose it as follows,

$$(2.1) \quad \mathfrak{m}^2 = \langle x^2 - x^3 \rangle \oplus \mathfrak{m}^3.$$

It is easy to see that the ideal $(x^2 - x^3)$ does not coincide with \mathfrak{m}^2 .

Below we follow [12] with some improvements.

As always we suppose that $S = R/I$, where $\mathfrak{m}^l \supset I \triangleleft R = \mathbb{K}[[x_1, \dots, x_n]]$ for $l \geq 2$. Assume that the algebra $\text{Der } S$ is not solvable. Hence it contains an \mathfrak{sl}_2 -triple $\{e, f, h\}$ with relations $[e, f] = h$, $[h, f] = -2f$, $[h, e] = 2e$. Note that the automorphisms of S preserve the maximal ideal $\bar{\mathfrak{m}} \triangleleft S$ as it is unique. All powers of the maximal ideal are preserved as well. Therefore the ideals $\bar{\mathfrak{m}}$ and $\bar{\mathfrak{m}}^2$ are \mathfrak{sl}_2 -submodules. Since the representations of \mathfrak{sl}_2 are completely reducible, the ideal $\bar{\mathfrak{m}}$ contains such an \mathfrak{sl}_2 -submodule \bar{V} that

$$(2.2) \quad \bar{\mathfrak{m}} = \bar{V} \oplus \bar{\mathfrak{m}}^2.$$

Denote by $\varphi: R \rightarrow S$ the factorization by the ideal I . Since $\varphi(\mathfrak{m}^2) = \bar{\mathfrak{m}}^2$ there exists a subspace $V \subset \mathfrak{m}$ such that $\mathfrak{m} = V \oplus \mathfrak{m}^2$ and $\varphi: V \xrightarrow{\sim} \bar{V}$. Thus, according to Proposition 2.1

the subspace V generates the ideal \mathfrak{m} and hence the algebra R . So we may assume up to the change of coordinates that $V = \langle x_1, \dots, x_n \rangle$ and $\bar{V} = \langle \bar{x}_1, \dots, \bar{x}_n \rangle$, where $\bar{x}_i = \varphi(x_i)$.

We may introduce an \mathfrak{sl}_2 -representation on V by the given isomorphism and extend it to R . Note that the factorization map φ is a homomorphism of \mathfrak{sl}_2 -modules. Therefore the ideal $I \triangleleft R$ is an invariant subspace of the \mathfrak{sl}_2 -representation on R .

Given a *weight vector* z , i.e. an eigenvector of the operator $h \in \mathfrak{sl}_2$, denote its weight by $\text{wt}(z) \in \mathbb{Z}$. We may suppose that x_1, \dots, x_n are the weight vectors of the \mathfrak{sl}_2 -module V , and x_1, \dots, x_k , $k \leq n$, are the highest weight vectors with weights $n_i = \text{wt}(x_i)$, where $n_1 \geq \dots \geq n_k \geq 0$, $\sum(n_i + 1) = n$. Denote $V_{\text{high}} := \langle x_1, \dots, x_k \rangle$, $V_{\text{rest}} := \langle x_{k+1}, \dots, x_n \rangle$.

The ideal $\mathfrak{m}I \subset R$ is \mathfrak{sl}_2 -invariant by the Leibniz rule, hence I contains the complementary \mathfrak{sl}_2 -submodule W such that $I = W \oplus \mathfrak{m}I$. By Corollary 2.2 its basis is a minimal set generating I .

Similarly to $V = V_{\text{high}} \oplus V_{\text{rest}}$ consider the decomposition

$$(2.3) \quad W = W_{\text{high}} \oplus W_{\text{rest}}$$

into the subspace $W_{\text{high}} = \langle w_1, \dots, w_s \rangle$, where w_i are the highest weight vectors of W , and the subspace W_{rest} of the remaining weight vectors of W . Notice that $W_{\text{rest}} \subset \text{Im } f \subset \langle x_{k+1}, \dots, x_n \rangle$ since $\text{Im } f$ is spanned by weight vectors which are not of highest weight.

Let $\varphi_i: R \rightarrow R/J_i$ be the factorization by the ideal $J_i = (x_{i+1}, \dots, x_n)$, $i = 1 \dots, k$. Since $J_i \supset (x_{k+1}, \dots, x_n) \supset W_{\text{rest}}$ the equality $W_i := \varphi_i(W) = \varphi_i(W_{\text{high}})$ holds. Note that $\dim W_i \geq i$, because $\mathbb{K}[[x_1, \dots, x_i]]/(W_i) \cong R/(J_i, W_i) \cong S/(\bar{x}_{i+1}, \dots, \bar{x}_n)$ is finite-dimensional. In particular, $s \geq k$.

By induction we can reorder the highest weight vectors $w_1, \dots, w_s \in W_{\text{high}}$ so that $\varphi_i(w_1), \dots, \varphi_i(w_i)$ become linearly independent in W_i for all i . Then $\text{wt}(w_i) \geq ln_i$ since w_i contains the monomials in variables $\bar{x}_1, \dots, \bar{x}_i$ of degree at least l and $\text{wt}(x_j) = n_j \geq n_i$ for $j \leq i$.

Proof of Theorem 1.1. The deduction above implies that the subspace V contains a non-trivial \mathfrak{sl}_2 -submodule and that $n_1 > 0$. We have

$$(2.4) \quad \dim I/\mathfrak{m}I = \dim W \geq \sum_{i=1}^k (ln_i + 1) = (n_1 - 1)l + l + 1 + \sum_{i=2}^k (ln_i + 1) \geq$$

$$(n_1 - 1) + l + 1 + \sum_{i=2}^k (n_i + 1) = \sum_{i=1}^k (n_i + 1) + l - 1 = n + l - 1.$$

Thus, Theorem 1.1 is proved. □

Proposition 2.3. *There exists a natural mapping $\varphi: \text{Aut } S \rightarrow \text{Aut}(\text{gr } S)$ with a unipotent kernel.*

Proof. The ideals $\bar{\mathfrak{m}}^i$ are invariant under $\text{Aut } S$ for all i , since they are powers of the unique maximal ideal. Therefore, $\text{Aut } S$ naturally acts on $\bar{\mathfrak{m}}^i/\bar{\mathfrak{m}}^{i+1}$ for all i , hence it acts on $\text{gr } S$. We obtain a natural map $\varphi: \text{Aut } S \rightarrow \text{Aut}(\text{gr } S)$.

Take a basis of S which is consistent with the chain of subspaces $0 \subset \bar{\mathfrak{m}}^r \subset \dots \subset \bar{\mathfrak{m}} \subset S$. Consider an arbitrary operator $g \in \ker \varphi$. Then $g(z) \in z + \bar{\mathfrak{m}}^{i+1}$ for any $z \in \bar{\mathfrak{m}}^i$, and g is represented by a unitriangular matrix in the taken basis. Hence $\ker \varphi$ is unipotent. \square

Corollary 2.4. *If the unity component $(\text{Aut}(\text{gr } S))^\circ$ is solvable then the unity component $(\text{Aut } S)^\circ$ is solvable as well.*

Theorem 2.5. *The algebra of derivations $\text{Der } S$ of a narrow algebra S is solvable.*

Proof. Let $\text{Der } S$ be non-solvable. Then the deduction above is applicable. Consider a simple \mathfrak{sl}_2 -submodule $F = \mathfrak{sl}_2 \cdot w_1 \subset I$. It has a zero intersection with the ideal $\mathfrak{m}I$. Let k be the biggest integer such that $F \subset \mathfrak{m}^k$. Then F has a zero intersection with \mathfrak{m}^{k+1} as well and the highest weight of F is equal to $kn_1 \geq k$. After factorization by \mathfrak{m}^{k+1} it implies that $\dim I_k \geq \dim(\mathfrak{m}I)_k + \dim F > \dim(\mathfrak{m}I)_k + k$, and the algebra S is not narrow. \square

Proof of Theorem 1.12. The desired statement is a direct consequence of Theorem 2.5 and Corollary 2.4. \square

Remark 2.6. As a matter of fact, Theorem 1.12 can be obtained without Proposition 2.3. However, then the proof loses in clarity.

Example 2.7. Algebras

$$(2.5) \quad A = \mathbb{K}[x, y]/(x^2, y^3, xy^2),$$

$$(2.6) \quad B = \mathbb{K}[x, y, z]/(x^3, x^2y, x^2z, y^4, z^4)$$

are extremal and Schulze's criterion is not applicable, but their algebras of derivations are solvable due to Theorem 1.12. Actually, the complete automorphism groups of A and B are solvable as well. For example, we obtain through a direct calculation

$$\text{Aut } A = \left\{ \begin{array}{l} \bar{x} \mapsto c_1\bar{x} + a_2\bar{x}\bar{y} + a_3\bar{y}^2, \\ \bar{y} \mapsto c_2\bar{y} + a_4\bar{x} + a_5\bar{x}\bar{y} + a_6\bar{y}^2 \end{array} \middle| a_i \in \mathbb{K}, c_i \in \mathbb{K}^\times \right\}.$$

Example 2.8. On the contrary, for the algebra

$$(2.7) \quad A = \mathbb{K}[x_1, \dots, x_n]/(x_1^l, x_2^l, \dots, x_n^l), \text{ where } l < n,$$

Schulze's criterion holds but the criterion of Theorem 1.12 does not. Note that for $n \geq 5$ the group $\text{Aut } A$ is non-solvable, as far as it contains the subgroup of permutations of coordinates.

Thereby, these two criteria have distinct areas of application.

3. EXTREMAL ALGEBRAS AND YAU'S THEOREM

Recall that by an extremal algebra we mean the finite-dimensional algebra satisfying the equality $\dim I/\mathfrak{m}I = l + n - 1$ in terms of Theorem 1.1.

Theorem 3.1. *An extremal algebra S has a non-solvable algebra of derivations $\text{Der } S$ if and only if it is of the form $S = S_1 \otimes S_2$, where*

$$(3.1) \quad S_1 \cong \mathbb{K}\llbracket x_1, x_2 \rrbracket / (x_1^l, x_1^{l-1}x_2, \dots, x_1x_2^{l-1}, x_2^l) \text{ for some } l \geq 2,$$

$$(3.2) \quad S_2 \cong \mathbb{K}\llbracket x_3, \dots, x_n \rrbracket / (w_2, \dots, w_{n-1}),$$

and $w_i \in \mathfrak{m}^l \cap \mathbb{K}\llbracket x_3, \dots, x_n \rrbracket$ form a regular sequence.

Proof. Suppose that $S = S_1 \otimes S_2$ as above. Then the group $\text{GL}(\langle x_1, x_2 \rangle)$ may be embedded into $\text{Aut } S_1$, hence the subalgebra S_1 carries a natural \mathfrak{sl}_2 -representation. We suppose this representation to be trivial on S_2 .

Vice-versa, let the algebra of derivations $\text{Der } S$ of the local algebra $S = R/I$ be non-solvable. Recall the deduction from Section 2. Then S is extremal if and only if the equalities hold in the chain of inequalities (2.4). The first equality holds if and only if W contains exactly k simple \mathfrak{sl}_2 -submodules, and their weights are ln_1, \dots, ln_k . The second inequality holds if and only if $n_1 = 1, n_2 = \dots = n_k = 0$.

Under these circumstances $k = n - 1$, and the simple \mathfrak{sl}_2 -submodules of \bar{V} are $\langle \bar{x}_1, \bar{x}_2 \rangle, \langle \bar{x}_3 \rangle, \dots, \langle \bar{x}_n \rangle$. Then $W_{\text{high}} = \langle w_1, \dots, w_{n-1} \rangle$, where $\text{wt}(w_1) = l, \text{wt}(w_i) = 0$ for $i = 2, \dots, n - 1$. We have

$$(3.3) \quad S = R / (w_1, f \cdot w_1, \dots, f^l \cdot w_1, w_2, \dots, w_{n-1}).$$

Note that the algebra \mathfrak{sl}_2 annihilates the series w_2, \dots, w_{n-1} , hence they do not depend on x_1 and x_2 and belong to $\mathbb{K}\llbracket x_3, \dots, x_n \rrbracket \cap \mathfrak{m}^l$. Since w_1 is the highest vector of weight l it is of the form $x_1^l g$, where $g \in \mathbb{K}\llbracket x_3, \dots, x_n \rrbracket$. Then $f^k \cdot w_1 = x_1^{l-k} x_2^k g$.

Taking into accordance $x_1^r \in I$ the equality

$$(3.4) \quad x_1^r = p_0 x_1^l g + p_1 x_1^{l-1} x_2 g + \dots + p_l x_2^l g + q_2 w_2 + \dots + q_{n-1} w_{n-1}$$

holds for certain $p_i, q_j \in R$. If we substitute x_1 for x_2 on the right side of equation, we get

$$(3.5) \quad x_1^r = (\tilde{p}_0 + \dots + \tilde{p}_l) x_1^l g + \tilde{q}_2 w_2 + \dots + \tilde{q}_{n-1} w_{n-1},$$

where $\tilde{p}_i = p_i(x_1, x_1, x_3, \dots, x_n)$, $\tilde{q}_j = q_j(x_1, x_1, x_3, \dots, x_n)$. Clearly, the series \tilde{q}_j and $\tilde{p} = \sum_{i=0}^l \tilde{p}_i$ may be assumed homogeneous in x_1 . Then $\tilde{p} = x_1^{r-l} \hat{p}$, $\tilde{q}_j = x_1^r \hat{q}_j$, where $\hat{p}, \hat{q}_j \in \mathbb{K}\llbracket x_3, \dots, x_n \rrbracket$. But it implies

$$(3.6) \quad x_1^l = \hat{p} x_1^l g + x_1^l \hat{q}_2 w_2 + \dots + x_1^l \hat{q}_{n-1} w_{n-1}.$$

Thus we can take the \mathfrak{sl}_2 -submodule $\langle x_1^l, x_1^{l-1} x_2, \dots, x_1 x_2^{l-1}, x_2^l \rangle$ instead of the \mathfrak{sl}_2 -submodule $\langle w_1, f \cdot w_1, \dots, f^l \cdot w_1 \rangle$ in (3.3).

Finally, the algebra S decomposes into the tensor product $S_1 \otimes S_2$, where S_i are as required. \square

Now let us introduce the following well-known technical lemma for power series, e.g. see [2, Section 11.1]. For an arbitrary power series g denote by $g_{(k)}$ its k th homogeneous component.

Lemma 3.2. *Suppose $p \in \mathfrak{m}^2 \setminus \mathfrak{m}^3 \subset R$. Then up to the analytical change of coordinates $p = x_1^2 + \dots + x_k^2 + q(x_{k+1}, \dots, x_n)$, where $q \in \mathfrak{m}^3 \cap \mathbb{K}\llbracket x_{k+1}, \dots, x_n \rrbracket$.*

Proof. The homogeneous component $p_{(2)}$ is a quadratic form, hence it is of the form $x_1^2 + \dots + x_k^2$ up to the linear change of coordinates. Consider a decomposition in x_1 as follows,

$$(3.7) \quad p = a_0 + a_1x_1 + a_2x_1^2 + \dots,$$

where $a_i \in \mathbb{K}[[x_2, \dots, x_n]]$ and a_2 is invertible.

Now consider a change of coordinates $\varphi : x_1 \mapsto x_1 + g, x_2 \mapsto x_2, \dots, x_n \mapsto x_n$, where $g \in \mathfrak{m} \cap \mathbb{K}[[x_2, \dots, x_n]]$. Thus,

$$(3.8) \quad \varphi(p) = \tilde{a}_0 + \tilde{a}_1x_1 + \tilde{a}_2x_1^2 + \dots$$

for some $\tilde{a}_i \in \mathbb{K}[[x_2, \dots, x_n]]$. In this case

$$(3.9) \quad \tilde{a}_1 = a_1 + 2ga_2 + 3g^2a_3 + \dots = a_2\left(\frac{a_1}{a_2} + 2g + 3g^2\frac{a_3}{a_2} + \dots\right).$$

Since $a_1 \in \mathfrak{m}$, we can choose such g that $\tilde{a}_1 = 0$. Indeed, let us divide a series in parentheses on the right side of equation (3.9) into homogeneous components. They are of the form $2g_{(k)} + P_k$, where P_k depends only on first $k - 1$ homogeneous components of g . Thus, by induction all the homogeneous components of the series g are uniquely defined. Note that $\tilde{a}_2 = a_2 + 3a_3g + 6a_4g^2 + \dots$ is still invertible.

Therefore we may suppose that $a_1 = 0$. Consider a change $x_1 \mapsto x_1b$ where $b^2 = a_2^{-1}$ (other coordinates are untouched). It is easy to see that the required series b exists. It is a change of coordinates since b is invertible. So, we may as well suppose $a_2 = 1$.

Finally, consider a change $x_1 \mapsto x_1 + b_2x_1^2 + b_3x_1^3 + \dots$. Similarly to the choice of g , we may take such b_i that $p \mapsto a_0 + x_1^2$. By induction by n we may suppose that $a_0(x_2, \dots, x_n)$ is of a required form. Then p is of a required form as well. \square

Proof of Theorem 1.5. According to Yau's remark in [16] we may suppose $p \in \mathfrak{m}^3$. Indeed, let $p \in \mathfrak{m}^2 \setminus \mathfrak{m}^3$. By Lemma 3.2 we have $p = x_1^2 + \dots + x_k^2 + q(x_{k+1}, \dots, x_n)$ up to the change of coordinates. Since $\frac{\partial p}{\partial x_i} = 2x_i, i = 1, \dots, k$, the equality $\mathbb{K}[[x_1, \dots, x_n]]/(p, J(p)) \cong \mathbb{K}[[x_{k+1}, \dots, x_n]]/(q, J(q))$ holds, and we may take a series $q \in \mathfrak{m}^3 \cap \mathbb{K}[[x_{k+1}, \dots, x_n]]$ instead of p .

If $p \in \mathfrak{m}^3$ then $I = (p, J(p)) \subset \mathfrak{m}^2$ and $l \geq 2$. We should prove that the Yau algebra $\text{Der } S$ of the moduli algebra $S = \mathbb{K}[[x_1, \dots, x_n]]/I$ is solvable. Assume the contrary.

Note that $\dim(I/\mathfrak{m}I) \leq n + 1 \leq n + l - 1$. Hence the moduli algebra either satisfy the inequality of Schulze's criterion and $\text{Der } S$ is solvable, or it is extremal and $l = 2$. In the latter case Theorem 3.1 may be applied and $S = S_1 \otimes S_2$, where $S_1 = \mathbb{K}[[x_1, x_2]]/(x_1^2, x_1x_2, x_2^2)$ and $S_2 = \mathbb{K}[[x_3, \dots, x_n]]/(w_2, \dots, w_{n-1})$.

It is easy to see that the homogeneous component $p_{(3)}$ has a form $p_1(x_1, x_2) + p_2(x_3, \dots, x_n)$, since otherwise $J(p)$ would contain a series with a term of the form x_ix_j , where $i \in \{1, 2\}, j \in \{3, \dots, n\}$. Therefore

$$(3.10) \quad \langle x_1^2, x_1x_2, x_2^2 \rangle \subset (p, J(p)) \cap (\mathbb{K}[[x_1, x_2]])_2 \subset \left\langle \frac{\partial p_1}{\partial x_1}, \frac{\partial p_1}{\partial x_2} \right\rangle,$$

a contradiction. \square

4. THE GLOBAL CASE AND HALPERIN'S CONJECTURE

Let the algebra $S = \mathbb{K}[x_1, \dots, x_n]/I$ be a finite-dimensional, not necessarily local algebra. Suppose that it contains s maximal ideals $\bar{\mathfrak{m}}_1, \dots, \bar{\mathfrak{m}}_s$. Then there exists an integer $k \in \mathbb{N}$ such that $\prod_{i=1}^s \bar{\mathfrak{m}}_i^k = 0$. Since the ideals $\bar{\mathfrak{m}}_i^k$ are coprime, the equation $\bigcap_{i=1}^s \bar{\mathfrak{m}}_i^k = \prod_{i=1}^s \bar{\mathfrak{m}}_i^k$ holds. Finally, by [4, Theorem 8.7] the algebra S can be decomposed into a direct product of local subalgebras as follows,

$$(4.1) \quad S \cong \prod_{i=1}^s S/\bar{\mathfrak{m}}_i^k.$$

In addition, any decomposition of S into the local subalgebras is of the form (4.1), i.e. they are uniquely determined up to isomorphism.

On the other hand, there is a unique maximal decomposition

$$(4.2) \quad 1 = e_1 + \dots + e_t$$

of the unity into a sum of orthogonal idempotents, i.e. $e_i e_i = e_i$ for all i and $e_i e_j = 0$ for $i \neq j$; e.g. see [5, Section II.5]. Then we have the decomposition

$$(4.3) \quad S = \bigoplus_{i=1}^t e_i S,$$

where the subalgebras $e_i S$ are indecomposable and hence local. It means that

$$(4.4) \quad S_i = e_i S \cong S/\bar{\mathfrak{m}}_i^k$$

up to the permutation of indexes and that $t = s$. Thus, we have a uniquely determined decomposition (4.3) of S into the local subalgebras.

Proposition 4.1. $(\text{Aut } S)^\circ = (\text{Aut } S_1)^\circ \times \dots \times (\text{Aut } S_s)^\circ$.

Proof. Since the unity 1 of algebra S and its decomposition (4.2) are unique, the primary idempotents e_i in this decomposition are preserved by $(\text{Aut } S)^\circ$ as well as the subalgebras S_i . Since $S_i S_j = 0$ for $i \neq j$, the required statement holds. \square

Denote by $\mathfrak{m}_i = (x_1 - a_{1i}, \dots, x_n - a_{ni})$, $i = 1 \dots s$, the maximal ideals in $\mathbb{K}[x_1, \dots, x_n]$ corresponding to $\bar{\mathfrak{m}}_1, \dots, \bar{\mathfrak{m}}_s \triangleleft S$. Then $\bigcap_{i=1}^s \mathfrak{m}_i^k \subset I$. Let us introduce the corresponding algebras of formal power series $R_i = \mathbb{K}[[x_1 - a_{1i}, \dots, x_n - a_{ni}]]$. The ideal $(\mathfrak{m}_j) \triangleleft R_i$ coincides with R_i unless $i = j$. In latter case $(\mathfrak{m}_i) \triangleleft R_i$ is the maximal ideal which we denote by $\tilde{\mathfrak{m}}_i$. Therefore, the inclusion $\tilde{\mathfrak{m}}_i^k = \bigcap_{i=1}^s (\mathfrak{m}_i^k) \subset (I) \triangleleft R_i$ holds, and

$$(4.5) \quad S_i \cong S/\bar{\mathfrak{m}}_i^k \cong \mathbb{K}[x_1, \dots, x_n]/(I, \mathfrak{m}_i^k) \cong R_i/(I).$$

Taking into account Proposition 4.1, we obtain the following solvability criterion.

Theorem 4.2. *The unity component $(\text{Aut } S)^\circ$ for the finite-dimensional algebra $S = \mathbb{K}[x_1, \dots, x_n]/I$ with maximal ideals $\bar{\mathfrak{m}}_1, \dots, \bar{\mathfrak{m}}_s$ is solvable if and only if the unity component $(\text{Aut } S_i)^\circ$ for the local algebra $S_i = R_i/(I)$ is solvable for each $i = 1 \dots s$.*

Now we can prove Halperin's conjecture.

Proof of Conjecture 1.2. Suppose $S = \mathbb{K}[x_1, \dots, x_n]/(f_1, \dots, f_n)$. Since the local subalgebra $S_i = R_i/(f_1, \dots, f_n)$ is a local complete intersection, the group $(\text{Aut } S_i)^\circ$ is solvable by Corollary 1.4 for each i . Then by Theorem 4.2 the group $(\text{Aut } S)^\circ$ is solvable as well. \square

Theorem 4.2 allows us to deduce the following globalization of Schulze's criterion.

Corollary 4.3. *Suppose the ideal $I \subset \mathbb{K}[x_1, \dots, x_n]$ with m generators and an integer $l > 1$ be such that the following holds.*

- *The quotient algebra $S = \mathbb{K}[x_1, \dots, x_n]/I$ is finite-dimensional.*
- *For any maximal ideal $\mathfrak{m} \subset \mathbb{K}[x_1, \dots, x_n]$ there holds either $I \not\subset \mathfrak{m}$ or $I \subset \mathfrak{m}^l$.*
- *An inequality $m < n + l - 1$ holds.*

Then the unity component $(\text{Aut } S)^\circ$ is solvable.

Proof. Let $\mathfrak{m}_1, \dots, \mathfrak{m}_s \triangleleft \mathbb{K}[x_1, \dots, x_n]$ be the only ideals containing I , as above. By Corollary 2.2 there holds $\dim(I/\mathfrak{m}_i I) \leq m < n + l - 1$ and Schulze's criterion is applicable for the algebras $R_i/(I)$, $i = 1, \dots, s$. Therefore $(\text{Aut } S)^\circ$ is solvable by Theorem 4.2. \square

5. AUTOMORPHISM SUBGROUPS AND DIMENSION BOUNDS

As usual we assume that the ideal $I \triangleleft R$ contains \mathfrak{m}^l , where $l \geq 2$, and the algebra $S = R/I$ is finite-dimensional and local with the maximal ideal $\bar{\mathfrak{m}} = (\bar{x}_1, \dots, \bar{x}_n)$.

Recall that the sum of all minimal ideals of a finite-dimensional algebra S is called a *socle* $\text{Soc } S$. It is invariant under endomorphisms of S . An *annihilator* of an arbitrary subset $X \subset S$ is the ideal $\text{Ann } X = \{z \in S \mid zX = 0\}$.

Lemma 5.1. $\text{Soc } S = \text{Ann } \bar{\mathfrak{m}}$.

Proof. Consider an arbitrary minimal ideal $J \subset \text{Soc } S$. Obviously, $\bar{\mathfrak{m}}J \subset J$. But $\bar{\mathfrak{m}}J \neq J$ by Nakayama's Lemma. Thus, $\bar{\mathfrak{m}}J = 0$ and $\text{Soc } S \subset \text{Ann } \bar{\mathfrak{m}}$.

Suppose $z \in \text{Ann } \bar{\mathfrak{m}}$. Then $zS = \{z(c + w) \mid c \in \mathbb{K}, w \in \bar{\mathfrak{m}}\} = \{cz \mid c \in \mathbb{K}\}$, so the principal ideal (z) is one-dimensional and minimal. It implies $\text{Ann } \bar{\mathfrak{m}} \subset \text{Soc } S$. \square

Assuming that S is graded, Y.-J. Xu and S. S.-T. Yau found a dimension bound for the group $\text{Aut } S$ as follows,

$$(5.1) \quad \dim \text{Aut } S \geq \dim S - \dim \text{Soc } S,$$

see [14, Proposition 2.3]. In Theorem 5.4 we introduce a lower bound without this assumption.

Definition 5.2. Let us call a *lower socle* of the algebra S the ideal $\text{LSoc } S = \text{Soc } S \cap \bar{\mathfrak{m}}^2$. We may choose a subspace $\text{USoc } S \subset \text{Soc } S$ such that

$$(5.2) \quad \text{Soc } S = \text{USoc } S \oplus \text{LSoc } S.$$

Let us call it an *upper socle*. Note that the choice of the upper socle is not canonical. However, up to the change of coordinates we may suppose $\text{USoc } S \subset \langle \bar{x}_1, \dots, \bar{x}_n \rangle$.

Proposition 5.3. *The automorphism group of a finite-dimensional local algebra S contains a unipotent subgroup $U \subset \text{Aut } S$ with*

$$(5.3) \quad \dim U = \dim(\text{LSoc } S) \cdot \dim(\bar{\mathfrak{m}}/\bar{\mathfrak{m}}^2) + \dim(\text{USoc } S) \cdot (\dim(\bar{\mathfrak{m}}/\bar{\mathfrak{m}}^2) - \dim(\text{USoc } S)).$$

Proof. Suppose that $\text{USoc } S = \langle \bar{x}_1, \dots, \bar{x}_s \rangle$. Consider the unipotent subgroup of linear transformations

$$(5.4) \quad U = \{u: \bar{x}_1 \mapsto \bar{x}_1 + F_1, \dots, \bar{x}_n \mapsto \bar{x}_n + F_n \mid F_1, \dots, F_s \in \text{LSoc } S, F_{s+1}, \dots, F_n \in \text{Soc } S\} \subset \text{GL}(S),$$

acting trivially on a subspace $\langle 1 \rangle \oplus \bar{\mathfrak{m}}^2$. It is easy to see that

$$(5.5) \quad u(\bar{x}_i)u(\bar{x}_j) = (\bar{x}_i + F_i)(\bar{x}_j + F_j) = \bar{x}_i\bar{x}_j = u(\bar{x}_i\bar{x}_j) \text{ for } i, j \in \{1, \dots, n\}, u \in U.$$

Hence

$$(5.6) \quad u(a)u(b) = ab = u(ab) \text{ for } a, b \in \bar{\mathfrak{m}}, u \in U.$$

Therefore the U -action is consistent with the multiplication in S , so $U \subset \text{Aut } S$. Finally,

$$(5.7) \quad \dim U = s \cdot \dim(\text{Soc } S) + (n - s) \cdot \dim(\text{LSoc } S) = s \cdot \dim(\text{USoc } S) + n \cdot \dim(\text{LSoc } S).$$

□

Theorem 5.4. $\dim \text{Aut } S \geq \dim(\bar{\mathfrak{m}}/\bar{\mathfrak{m}}^2) \cdot \dim \text{Soc } S$.

Proof. Consider a subgroup $G = \text{GL}(\text{USoc } S) \subset \text{Aut } S$. Along with the subgroup U from Proposition 5.3 it generates a subgroup $GU \subset \text{Aut } S$. To prove the inequality

$$(5.8) \quad \dim GU \geq \dim G + \dim U$$

it suffices to look at the tangent algebras of G and U . Indeed, easy to see that they have a zero intersection, and

$$(5.9) \quad \begin{aligned} \dim GU &= \dim(\text{Lie } GU) \geq \dim(\text{Lie } G) + \dim(\text{Lie } U) = \\ &= \dim G + \dim U = (\dim(\text{USoc } S))^2 + \dim(\text{LSoc } S) \cdot \dim(\bar{\mathfrak{m}}/\bar{\mathfrak{m}}^2) + \\ &= \dim(\text{USoc } S) \cdot (\dim(\bar{\mathfrak{m}}/\bar{\mathfrak{m}}^2) - \dim(\text{USoc } S)) = \dim(\text{Soc } S) \cdot \dim(\bar{\mathfrak{m}}/\bar{\mathfrak{m}}^2). \end{aligned}$$

□

Corollary 5.5. *The group $\text{Aut } S$ is infinite if $S \neq \mathbb{K}$.*

Clearly, the automorphism group almost always contains a rather big unipotent subgroup. A natural question arises if the whole automorphism group may be unipotent. The following proposition provides an example.

Proposition 5.6. *Consider the following local algebra*

$$(5.10) \quad S = \mathbb{K}[x, y]/I, \quad I = (y^5, (x + y)^6, x^5 - x^3y^3, x^4y).$$

Then the group $\text{Aut } S$ is unipotent.

Proof. The Gröbner basis of the ideal I with respect to the homogeneous lexicographic order with $x \prec y$ is

$$(5.11) \quad \{x^6, y^5, x^3y^3 - x^5, 3x^2y^4 + 4x^5, x^4y\}.$$

Clearly, $\mathfrak{m}^7 \subset I$. Let \bar{x}, \bar{y} be the images of x, y respectively under factorization by I . The basis of algebra S is as follows.

$$\begin{array}{ccccccc} 1 & \bar{x} & \bar{x}^2 & \bar{x}^3 & \bar{x}^4 & (\bar{x}^5) \\ \bar{y} & \bar{x}\bar{y} & \bar{x}^2\bar{y} & \bar{x}^3\bar{y} & & \\ \bar{y}^2 & \bar{x}\bar{y}^2 & \bar{x}^2\bar{y}^2 & \bar{x}^3\bar{y}^2 & & \\ \bar{y}^3 & \bar{x}\bar{y}^3 & \bar{x}^2\bar{y}^3 & (\bar{x}^3\bar{y}^3) & & \\ \bar{y}^4 & \bar{x}\bar{y}^4 & (\bar{x}^2\bar{y}^4) & & & \end{array}$$

where $-\frac{3}{4}\bar{x}^2\bar{y}^4 = \bar{x}^3\bar{y}^3 = \bar{x}^5$.

Consider an arbitrary automorphism $\varphi \in \text{Aut } S$,

$$(5.12) \quad \varphi(\bar{x}) = a_{11}\bar{x} + a_{12}\bar{y} + h_1(\bar{x}, \bar{y}), \quad h_1 \in \mathfrak{m}^2,$$

$$(5.13) \quad \varphi(\bar{y}) = a_{21}\bar{x} + a_{22}\bar{y} + h_2(\bar{x}, \bar{y}), \quad h_2 \in \mathfrak{m}^2.$$

Note that \bar{y} is the only linear polynomial whose 5th power degree is zero. Then $\varphi(\bar{y}^5) = 0$ implies $a_{21} = 0$. On the other hand, \bar{x} is the only linear polynomial whose 5th degree is not zero but lies in $\bar{\mathfrak{m}}^6$. Therefore,

$$(5.14) \quad \varphi(\bar{x}) = a_{11}\bar{x} + h_1(\bar{x}, \bar{y}), \quad h_1 \in \mathfrak{m}^2,$$

$$(5.15) \quad \varphi(\bar{y}) = a_{22}\bar{y} + h_2(\bar{x}, \bar{y}), \quad h_2 \in \mathfrak{m}^2,$$

where $a_{11}, a_{22} \neq 0$ due to invertibility of φ . Then we should notice that $\varphi((\bar{x} + \bar{y})^6) = 0$ if and only if $a_{11} = a_{22} = c$. Finally, $\varphi(x^3y^3 - x^5) = c^6x^3y^3 - c^5x^5 = 0$. It implies that $c = 1$.

Hence for an arbitrary element $z \in \bar{\mathfrak{m}}^i$ an inclusion $(\text{Aut } S) \cdot z \subset z + \bar{\mathfrak{m}}^{i+1}$ holds. Thus, the group $\text{Aut } S$ is unipotent. \square

6. ACKNOWLEDGEMENTS

The author would like to express his gratitude to his scientific advisor I. Arzhantsev for posing the problem, useful discussions and remarks. Thanks are also due to A. Elashvili for a thoughtful discussion and to M. Zaidenberg for constructive remarks.

REFERENCES

- [1] S.S. Abhyakar, *Local analytic geometry*, World Scientific Publishing Co. Inc., River Edge, NJ, 2001.
- [2] V.I. Arnold, S.M. Gusein-Zade, A.N. Varchenko, *Singularities of differentiable maps, Vol. 1*, Monographs in Math. **82**, Birkhäuser, Boston, 1985.
- [3] I.V. Azhantsev, D.A. Timashev, *On the canonical embedding of certain homogeneous spaces*. In: “Lie Groups and Invariant Theory: A.L. Onishchik’s jubilee volume” (E.B. Vinberg, Editor), AMS Translations, Series 2, vol. **213** (2005), 63–83.
- [4] M.F. Atiyah, I.G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley Publishing, Massachusetts, 1969.
- [5] M. Auslander, I. Reiten, S.O. Smalø, *Representation theory of Artin algebras*, Cambridge Stud. Adv. Math. **36**, Cambridge Univ. Press, 1995.

- [6] A. Elashvili, G. Khimshiashvili, *Lie Algebras of Simple Hypersurface Singularities*, J. Lie Theory **16** (2006), 621–649.
- [7] T. de Jong, G. Pfister, *Local Analytic Geometry: Basic Theory and Applications*, Advanced Lectures in Mathematics, Vieweg, Braunschweig, 2000.
- [8] G.R. Kempf, *Jacobians and Invariants*, Invent. Math. **112** (1993), 315–321.
- [9] H. Kraft, C. Procesi, *Graded morphisms of G -modules*, Ann. Inst. Fourier **37** (1987), no. 4, 161–166.
- [10] J. Mather, S. S.-T. Yau, *Classification of isolated hypersurface singularities by their moduli algebras*, Invent. Math. **69** (1982), 243–251.
- [11] A.L. Onishchik, E.B. Vinberg, *Lie Groups and Algebraic Groups*, Springer–Verlag, Berlin, 1990.
- [12] M. Schulze, *A solvability criterion for the Lie algebra of derivations of a fat point*, J. Algebra **323** (2010), no. 10, 2916–2921.
- [13] V. Weispfenning, *Two Model Theoretic Proofs of Rückert’s Nullstellensatz*, Trans. AMS **203** (1975), 331–342.
- [14] Y.-J. Xu, S. S.-T. Yau, *Micro-Local Characterization of Quasi-Homogeneous Varieties*, Amer. J. Math. **118** (1996), no. 2, 389–399.
- [15] S. S.-T. Yau, *Continuous family of finite dimensional representations of a solvable Lie algebra arising from singularities*, Proc. Nat. Acad. Sci. USA **80** (Dec. 1983), 7694–7696.
- [16] S. S.-T. Yau, *Solvability of Lie algebras arising from isolated singularities and nonisolatedness of singularities defined by $sl(2, \mathbb{C})$ invariant polynomials*, Amer. J. Math. **113** (1991), no. 5, 773–778.

E-mail address: perepechko@mccme.ru

DEPARTMENT OF HIGHER ALGEBRA, FACULTY OF MECHANICS AND MATHEMATICS, MOSCOW STATE UNIVERSITY, LENINSKIE GORY, MOSCOW, 119991, RUSSIA

UNIVERSITÉ GRENOBLE I, INSTITUT FOURIER, UMR 5582 CNRS-UJF, BP 74, 38402 ST. MARTIN D’HÈRES CÉDEX, FRANCE